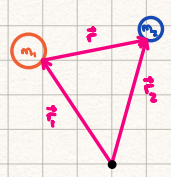
Geometric Algebra - Kepler Problem (Part 1)

8/9/19

* Overview of topics covered
  - Set up the problem
  - Two conserved quantities:
    ① Angular Momentum
    ② Energy

* Set up problem:
  - Kepler Problem: What is the motion of two bodies due to gravity?
  - Consider two masses, m_1 & m_2 and two displacement vectors $\vec{r}_1$ & $\vec{r}_2$ that exist btwn a point in space and the two respective masses



  - Let $\vec{r}$ be some relative displacement vector defined as $\vec{r} = \vec{r}_2 - \vec{r}_1$
  - Suppose m_1 represents the Sun and m_2 represents the Earth
    ⇒ $\vec{r}$ is the vector pointing from the Sun to the Earth
    • One of our goals is to describe the evolution of $\vec{r}$
  - Now, let's use Newton's 2nd Law to describe how m_1 & m_2 accelerate (i.e., the 2nd time derivatives of $\vec{r}_1$ & $\vec{r}_2$)

$$m_1 = \frac{\vec{F}_{12}}{\ddot{\vec{r}}_1} \Rightarrow \ddot{\vec{r}}_1 = \frac{\vec{F}_{12}}{m_1}$$

$$m_2 = \frac{\vec{F}_{21}}{\ddot{\vec{r}}_2} \Rightarrow \ddot{\vec{r}}_2 = \frac{\vec{F}_{21}}{m_2}$$

  - Applying Newton's 3rd Law gives $\vec{F}_{12} = -\vec{F}_{21}$
  - The direction & magnitude of $\vec{F}_{12}$ (i.e., the force of gravity) is given by the universal gravitational law
    • The direction of $\vec{F}_{12}$ is going to be oriented along the line joining m_1 & m_2
  - Let $\hat{r}$ denote the unit vector pointing in the same direction as $\vec{r}$
    • Applying the universal law of gravitation allows us to write $\vec{F}_g$ in terms of $\hat{r}$ as follows:
      * Note: $\hat{r}$, $\vec{r}$, and $\vec{F}_{12}$ all point in the same direction

$$\vec{F}_{12} = \frac{G m_1 m_2}{r^2}\hat{r} \quad \text{where } r = |\vec{r}| \text{ and } G \text{ is the universal gravitational constant}$$

    • Combine this expression w/ our definition for $\ddot{\vec{r}}_1$:

$$\ddot{\vec{r}}_1 = \frac{\vec{F}_{12}}{m_1} = \frac{\frac{G m_1 m_2}{r^2}\hat{r}}{m_1} = \frac{G m_2}{r^2}\hat{r}$$

      ⇒ Thus, our acceleration vector for m_1 points in the same direction as $\hat{r}$
    ~ Applying $\vec{F}_{12} = -\vec{F}_{21}$ and our def'n for $\ddot{\vec{r}}_2$ gives...

$$\ddot{\vec{r}}_2 = -\frac{G m_2}{r^2}\hat{r}$$

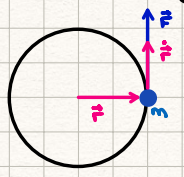
  - It turns out that $\ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 = \frac{-G(m_2 + m_1)}{r^2}\hat{r}$
  - Let $\mu = G(m_2 + m_1)$ denote the sum of gravitational parameters in our system and can be treated as a constant

$$\Rightarrow \boxed{\ddot{\vec{r}} = -\frac{\mu}{r^2}\hat{r}}$$

* This is the primary eqn. we are going to be using over the next few videos
  ⇒ Tells us that $\ddot{\vec{r}}$ is the negative of some scalar multiplied by $\hat{r}$
    ⇒ i.e., $\vec{r}$ accelerates in the direction opposite that of $\hat{r}$ (makes sense from the PoV of m_1 since "all" the forces acting on m_1 point in the opposite direction as $\hat{r}$ ∴ $\ddot{\vec{r}}$ should also point in the direction opposite that of $\hat{r}$)

Aside on Angular Momentum

- "Typical" view of angular momentum
  - Consider a point mass m rotating about some origin along a perfectly circular path due to some tangential force



$\vec{L} = \vec{r} \times m\dot{\vec{r}}$ angular momentum for a point mass m travelling along a circular path w/ radius $\vec{r}$

#Note: Here, the right hand rule tells us $\vec{L}$ points directly out of the page (assuming our circular path lies along the plane of the page)

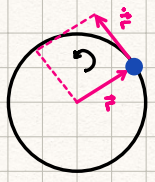
  - ~Also, we can write this in terms of linear momentum by substituting in $\vec{p} = m\dot{\vec{r}}$:

$$\vec{L} = \vec{r} \times \vec{p}$$

- Now, let's define angular momentum in terms of geometric algebra
  - Let H denote angular momentum per unit mass

$\Rightarrow$ can write $\boxed{H = \vec{r} \wedge \dot{\vec{r}}}$

~Diagram:



* Importantly, this way of defining angular momentum carries all the same info. as the convential formulation (also doesn't require we "invoke" a 3rd dimension orthogonal to the plane along which our circular path is defined)

#Now, let's try to implement $H = \vec{r} \wedge \dot{\vec{r}}$ into our gravitational problem

- First, let's evaluate the first time derivative for H

$$\frac{dH}{dt} = \frac{d}{dt}(\vec{r} \wedge \dot{\vec{r}})$$

  - Apply the product rule:

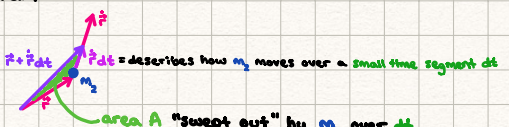
$$\frac{dH}{dt} = \dot{\vec{r}} \wedge \vec{r} + \vec{r} \wedge \ddot{\vec{r}} = \vec{r} \wedge \ddot{\vec{r}}$$

  - ~Since $\vec{r}$ and $\ddot{\vec{r}}$ have orientations parallel to each other, $\vec{r} \wedge \ddot{\vec{r}} = 0$

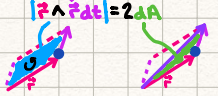
$\Rightarrow \frac{dH}{dt} = \frac{d}{dt}(\vec{r} \wedge \dot{\vec{r}}) = 0$ $\therefore$ H is a quantity that is conserved over time

$\Rightarrow$ agrees w/ the concept of conservation of angular momentum

- Now, let's combine conservation of angular momentum w/ Kepler's 2nd Law (Law of Equal Areas Over Equal Times)
  - Consider our relative displacement vector $\vec{r}$, mass m_2, and an infinitesimally small amount of time dt
    - Also, suppose that m_2 has some velocity $\dot{\vec{r}}$
    - ~Diagram:


$\vec{r} + \dot{\vec{r}}dt$
$\dot{\vec{r}}dt$ = describes how m_2 moves over a small time segment dt
$\dot{\vec{r}}$
m_2
$\vec{r}$
area A "swept out" by m_2 over dt

$\Rightarrow$ Note: The magnitude of $\vec{r} \wedge \dot{\vec{r}}dt$ is going to be the area of the parallelogram defined by $\vec{r}$ & $\dot{\vec{r}}dt$



$\Rightarrow$ $dA = \frac{1}{2}\vec{r} \wedge \dot{\vec{r}}dt$

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- Equation we left off with:

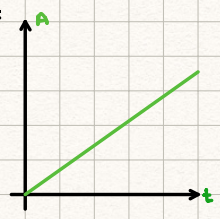
$d\vec{A} = \frac{1}{2}\vec{r} \wedge \dot{\vec{r}}\,dt$ where dA = an infinitely small change in area traversed by mass m_2 over an infinitely small time dt

$\Rightarrow \frac{dA}{dt} = \dot{A} = \frac{1}{2}\vec{r} \wedge \dot{\vec{r}}$

- Also recall that we defined $H = \vec{r} \wedge \dot{\vec{r}}$ as the angular momentum per unit mass

$\Rightarrow \dot{A} = \frac{1}{2}H = \text{constant}$

•Plot:



*This implies Kepler's 2nd Law => a mass will traverse along any orbital path s.t. the amount of infinitely small area "swept" by a mass over an infinitely small amount of time

=> The gravitational law gives orbitals in the shape of circles and/or ellipticals

*Rewrite $H = \vec{r} \wedge \dot{\vec{r}}$ in terms of $\hat{r}$ using $\hat{r} = \frac{\vec{r}}{r}$:

$\hat{r} = \frac{\vec{r}}{r} \Rightarrow \vec{r} = r\hat{r}$

$\Rightarrow H = (r\hat{r}) \wedge (\dot{r}\hat{r} + r\dot{\hat{r}}) = r\hat{r} \wedge \dot{r}\hat{r} + r\hat{r} \wedge r\dot{\hat{r}} = r\dot{r}\cancel{(\hat{r} \wedge \hat{r})}^{0} + r^2(\hat{r} \wedge \dot{\hat{r}})$

$\Rightarrow H = r^2(\hat{r} \wedge \dot{\hat{r}})$

*Recall: $\hat{r}\dot{\hat{r}} = \hat{r} \cdot \dot{\hat{r}} + \hat{r} \wedge \dot{\hat{r}}$

=> Substitute $\hat{r} \wedge \dot{\hat{r}} = \hat{r}\dot{\hat{r}} - \hat{r} \cdot \dot{\hat{r}}$ into expression for H:

$H = r^2(\hat{r}\dot{\hat{r}} - \hat{r} \cdot \dot{\hat{r}})$

•If we assume $\hat{r} \perp \dot{\hat{r}}$ (i.e., purely tangential motion wrt the orbital path), $\hat{r} \cdot \dot{\hat{r}} = 0$

~Aside: We can interpret the orthogonality relation btwn $\hat{r}$ & $\dot{\hat{r}}$ as follows (this will also be useful for conservation of energy later):

$\hat{r} \cdot \dot{\hat{r}} = \frac{1}{2}\frac{d}{dt}(\hat{r} \cdot \hat{r})$ => i.e., $\hat{r} \cdot \dot{\hat{r}}$ can be interpreted as one-half of the product rule for $\hat{r} \cdot \hat{r}$

~Recall: $\hat{r} \cdot \hat{r} = |\hat{r}|^2 = \text{constant}$ (since $\hat{r}$ is a constant)

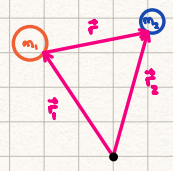
$\Rightarrow \hat{r} \cdot \dot{\hat{r}} = \frac{1}{2}\frac{d}{dt}(\text{constant}) = \frac{1}{2}(0) = 0$ ✓

=> End up w/ $H = r^2(\hat{r}\dot{\hat{r}} - \cancel{\hat{r} \cdot \dot{\hat{r}}}^{0}) = r^2\hat{r}\dot{\hat{r}}$

$\Rightarrow H = \underbrace{r^2}_{\text{scalar}}\underbrace{\hat{r}\dot{\hat{r}}}_{\text{bivector}}$ (angular momentum per unit mass)

*Note: H is a bivector

* Overview of our system so far:



$$\ddot{\vec{r}} = -\frac{\mu}{r^2}\hat{r}$$

- Let's evaluate $\ddot{\vec{r}}$ "dotted" w/ $\dot{\vec{r}}$:

$$\ddot{\vec{r}}\cdot\dot{\vec{r}} = -\frac{\mu}{r^2}\hat{r}\cdot\dot{\vec{r}}$$

- LHS: Similar to the way we treated $\hat{r}\cdot\dot{\hat{r}}$ as one-half of the product rule applied to $\hat{r}\cdot\hat{r}$, we can write...

$$\ddot{\vec{r}}\cdot\dot{\vec{r}} = \frac{1}{2}\frac{d}{dt}\left(\dot{\vec{r}}\cdot\dot{\vec{r}}\right)$$

- RHS: According to the product rule, we can write $\dot{\vec{r}} = \dot{r}\hat{r} = \dot{r}\hat{r} + r\dot{\hat{r}}$

$$-\frac{\mu}{r^2}\hat{r}\cdot\dot{\vec{r}} = -\frac{\mu}{r^2}\hat{r}\cdot\left(\dot{r}\hat{r} + r\dot{\hat{r}}\right) = -\frac{\mu}{r^2}\left[\left(\hat{r}\cdot\dot{r}\hat{r}\right) + \left(\hat{r}\cdot r\dot{\hat{r}}\right)\right] = -\frac{\mu}{r^2}\left[\dot{r}\underbrace{\left(\hat{r}\cdot\hat{r}\right)}_{=|\hat{r}|^2} + r\underbrace{\left(\hat{r}\cdot\dot{\hat{r}}\right)}_{0\ (\text{last page, we specified } \hat{r}\perp\dot{\hat{r}}\ \therefore\ \hat{r}\cdot\dot{\hat{r}}=0)}\right]$$

~ If we assume $|\hat{r}| = 1$, we can write...

$$-\frac{\mu}{r^2}\hat{r}\cdot\dot{\vec{r}} = -\frac{\mu}{r^2}\dot{r}\underbrace{|\hat{r}|^2}_{1} = -\frac{\mu}{r^2}\dot{r}$$

- Rewrite LHS & RHS as one equation:

$$\frac{1}{2}\frac{d}{dt}\underbrace{\left(\dot{\vec{r}}\cdot\dot{\vec{r}}\right)}_{=|\dot{\vec{r}}|^2=v^2} = \underbrace{-\frac{\mu}{r^2}\dot{r}}_{=-\mu\frac{1}{r^2}\frac{dr}{dt}=\frac{d}{dt}\left(\frac{\mu}{r}\right)}$$

?! I'm missing something here

$$\Rightarrow \frac{d}{dt}\left(\frac{v^2}{2}\right) = \frac{d}{dt}\left(\frac{\mu}{r}\right) \Rightarrow \frac{d}{dt}\left(\frac{v^2}{2}\right) - \frac{d}{dt}\left(\frac{\mu}{r}\right) = 0 \Rightarrow \underbrace{\frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right)}_{\text{The rate of change of this quantity is constant over time}} = 0$$

=> however, v and r do change - just in a way s.t. this overall quantity stays constant over time

- As it turns out, $\frac{1}{2}v^2 - \frac{\mu}{r}$ has units in energy per unit mass

$$\Rightarrow \frac{1}{2}v^2 - \frac{\mu}{r} = E = \text{energy per unit mass}$$

- Each term corresponds w/ two different "components" of energy per unit mass
  - ~ Let $T = \frac{1}{2}v^2$ denote the kinetic energy per unit mass
  - ~ Let $U = -\frac{\mu}{r}$ denote the gravitational potential energy per unit mass

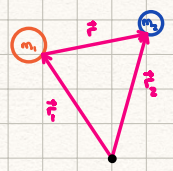
$$\Rightarrow \boxed{E = T + U}$$

* Take-Home-Message: T and U may change over time, but their sum is constant
  - ~ Also agrees w/ trade-off btwn the velocity of m_2 and the distance btwn m_1 and m_2

Geometric Algebra - Kepler Problem (Part 2)

8/13/19

* Overview of Topics Covered
  - Review of Last Video
  - Inverse-square law ⟶ conic sections
* Review of Last Video:



- $\vec{r}$ = relative displacement vector btwn m_1 & m_2
  - Note: $\vec{r} = \vec{r}_2 - \vec{r}_1$
- Also found...

$$\ddot{\vec{r}} = -\frac{Gm_1m_2}{r^2}\hat{r}$$ where $\hat{r}$ denotes a unit vector (i.e., $|\hat{r}|=1$) pointing in the same direction as $\vec{r}$

- Defined $\mu = Gm_1m_2$ as a "gravitational parameter"

  ⇒ Allowed us to write $\ddot{\vec{r}} = -\frac{\mu}{r^2}\hat{r}$

- Implemented some GA concepts including defining $H = \vec{r}\wedge\dot{\vec{r}}$ as the angular momentum per unit mass
  - It also turns out that...

    $\frac{d}{dt}(\vec{r}\wedge\dot{\vec{r}}) = 0$ (i.e., angular momentum per unit mass is a conserved quantity over time)
  - Ended up deriving an expression for the conservation of energy (per unit mass) for our system:

$$\frac{d}{dt}\underbrace{\left(\frac{1}{2}v^2 - \frac{\mu}{r^2}\right)}_{\text{energy per unit mass}} = 0$$

  ⇒ suggests a trade-off btwn velocity (i.e., kinetic energy per unit mass) and relative displacement (i.e., potential energy per unit mass)
- Importantly, we identified a conserved quantity that can be defined as a bivector (i.e., wedge product) and a conserved quantity that can be defined as a scalar (i.e., dot product)
  - Bivector is associated w/ angular momentum per unit mass, $H = \vec{r}\wedge\dot{\vec{r}}$
  - Scalar is associated w/ energy per unit mass, $E = \frac{1}{2}v^2 - \frac{\mu}{r^2}$
- Also derived $H = \vec{r}\wedge\dot{\vec{r}} = r^2\hat{r}\dot{\hat{r}}$ (assuming $\vec{r}\perp\dot{\vec{r}}$) where $\hat{r}\dot{\hat{r}}$ is the geometric product

* Now, lets multiply the following two equations:

$$\ddot{\vec{r}} = -\frac{\mu}{r^2}\hat{r} \quad \text{and} \quad H = r^2\hat{r}\dot{\hat{r}}$$

$$\Rightarrow H\ddot{\vec{r}} = (r^2\hat{r}\dot{\hat{r}})\left(-\frac{\mu}{r^2}\hat{r}\right) = -\mu\dot{\hat{r}}\hat{r}$$

- Since $\hat{r}\perp\dot{\hat{r}}$, their geometric product anticommutes: $\hat{r}\dot{\hat{r}} = \hat{r}\wedge\dot{\hat{r}} = -\dot{\hat{r}}\wedge\hat{r} = -\dot{\hat{r}}\hat{r}$

$$\Rightarrow H\ddot{\vec{r}} = -\mu\dot{\hat{r}}\hat{r} = -\mu(-\hat{r}\dot{\hat{r}}) = \mu\dot{\hat{r}}\hat{r}^2$$

- Since $|\hat{r}|=1$ and $\hat{r}^2 = \hat{r}\hat{r} = \hat{r}\cdot\hat{r} = |\hat{r}|^2$, $\hat{r}^2 = |\hat{r}|^2 = 1^2 = 1$

$$\Rightarrow H\ddot{\vec{r}} = \mu\dot{\hat{r}}\hat{r}^2 = \mu\dot{\hat{r}}(1) = \mu\dot{\hat{r}} \quad \Rightarrow H\ddot{\vec{r}} - \mu\dot{\hat{r}} = 0$$

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- Derived $H\ddot{\hat{r}} - \mu\dot{\hat{r}} = 0$

$\Rightarrow \frac{d}{dt}(H\dot{\hat{r}} - \mu\hat{r}) = 0$ (here, we make use of the fact that $\dot{H} = 0$)

* We end up w/ another conservation expression!

$\Rightarrow$ Here, the vector-quantity given by $H\dot{\hat{r}} - \mu\hat{r} = \vec{c}$ is constant over time

- Note: Scalar multiples of $\vec{c}$ are more formally known as Laplace-Runge-Lentz vectors

- Importantly, $\vec{c}$ is conserved in our Kepler problem

* Summary of Kepler Problem so far:

- Found three conserved quantities:

① Energy per unit mass, E; grade 0 quantity

② Laplace vector, $\vec{c}$; grade 1 quantity

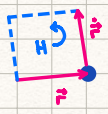
③ Angular momentum per unit mass, H; grade 2 quantity

- Aside: Multiplying H (a bivector) w/ $\dot{\hat{r}}$ (a vector) - i.e., the first term in $\frac{d}{dt}(H\dot{\hat{r}} - \mu\hat{r}) = 0$

~ Diagram for an arbitrary H "object":

* Recall: $H = r^2\hat{r}\dot{\hat{r}}$

$= r^2\hat{r}\wedge\dot{\hat{r}}$ (assuming $\hat{r} \perp \dot{\hat{r}}$)

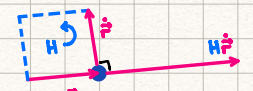


$\Rightarrow$ Recall: The general convention for multiplying a bivector (e.g., H) w/ a vector entails two cases:

a) Multiply the bivector on the RHS of the vector $\Rightarrow$ preserves orientation of rotation given by the bivector

b) Multiply the bivector on the LHS of the vector $\Rightarrow$ reverses orientation of rotation given by the bivector

$\Rightarrow$ $H\dot{\hat{r}}$ = the $\dot{\hat{r}}$ vector rotated 90° clockwise and scaled by a factor of r^2



* End result is a vector

$\Rightarrow$ Thus, our $H\dot{\hat{r}} - \mu\hat{r} = \vec{c}$ conservation expression can be described mathematically as the distance btwn two vector quantities, $H\dot{\hat{r}}$ and $\mu\hat{r}$

$\Rightarrow$ It turns out that our Laplace vector, $\vec{c}$ gives a "clean" soln. to the Kepler problem

* Let's start rearranging our conservation expression that defines $\vec{c}$:

$H\dot{\hat{r}} - \mu\hat{r} = \vec{c}$

$\Rightarrow H\dot{\hat{r}} = \vec{c} + \mu\hat{r}$

- Multiply both sides on the right w/ $\vec{r}$:

$H\dot{\hat{r}}\vec{r} = \vec{c}\vec{r} + \mu\hat{r}\vec{r}$

$\Rightarrow H(\dot{\hat{r}}\cdot\vec{r} + \dot{\hat{r}}\wedge\vec{r}) = (\vec{c}\cdot\vec{r} + \vec{c}\wedge\vec{r}) + \mu(\hat{r}\cdot\vec{r} + \hat{r}\wedge\vec{r})$ ($\hat{r}\wedge\vec{r} = 0$)

$\Rightarrow \underbrace{(\dot{\hat{r}}\cdot\vec{r})}_{=\text{scalar}}H + \underbrace{H(\dot{\hat{r}}\wedge\vec{r})}_{=-H} = \underbrace{(\vec{c}\cdot\vec{r})}_{=cr\cos\theta=\text{scalar}} + \underbrace{\vec{c}\wedge\vec{r}}_{=\text{bivector}} + \mu\underbrace{(\hat{r}\cdot\vec{r})}_{=(1)r\cos(0)=r=\text{scalar}}$ * Note: $c = |\vec{c}|$

$\Rightarrow \underbrace{(\dot{\hat{r}}\cdot\vec{r})}_{=\text{bivector}}H - \underbrace{H^2}_{=\text{scalar}} = \underbrace{cr\cos\theta + \mu r}_{=\text{scalar}} + \underbrace{\vec{c}\wedge\vec{r}}_{\text{bivector}}$

- Proof that H^2 returns a scalar quantity

~ Note: Numerically, we can write H as a scalar multiple of the basis bivector, e_1e_2 $\Rightarrow H = he_1e_2$

- Here, the scalar a represented the signed area associated w/ angular momentum

$\Rightarrow H^2 = h^2(e_1e_2)^2 = h^2I^2 = h^2(-1) = -h^2$

$\Rightarrow \boxed{H^2 = -h^2}$ * h = angular momentum per unit mass

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- Substitute $H^2 = -h^2$ into our conservation expression that contains the Lorentz factor, $\vec{c}$:

$$(\vec{r}\cdot\vec{r})\vec{H} + H^2 = r\left(c(\cos\theta) + \mu\right) + \vec{c}\times\vec{r}$$

$$\underbrace{(\vec{r}\cdot\vec{r})\vec{H}}_{\text{bivector}} - \underbrace{h^2}_{\text{scalar}} = \underbrace{r\left(c(\cos\theta) + \mu\right)}_{\text{scalar}} + \underbrace{\vec{c}\times\vec{r}}_{\text{bivector}}$$

- Set scalar terms equal to each other: $h^2 = r(c(\cos\theta) + \mu)$

  => Solve for r:

$$r = \frac{h^2}{\mu + c(\cos\theta)} = \frac{\frac{h^2}{\mu}}{1 + c(\cos\theta)}$$

  - Treating θ as a variable allows us to write r as a function of θ (i.e., r is written in polar coordinates)

Define some new constants:

$\rho = h^2/\mu$ and $e = c/\mu$

- It turns out that ρ and e are parameters of a conic section
  - ρ parameterizes the semi-latus rectum
  - e parameterizes the the eccentricity
- Writing our r eqn. in terms of ρ & e give...

$$\boxed{r = \frac{\rho}{1 + e\cos\theta}}$$ Claim: Eqns. of this form are conic sections

- Solve for ρ:

$$\rho = r + er\cos\theta$$

- Convert to Cartesian coordinates using $x = r\cos\theta$:

$$\rho = r + ex$$

- Rewrite eqn. as an expression for r:

$$r = \rho - ex$$

- Square both sides: $r^2 = (\rho - ex)^2$
  - Use the polar → Cartesian coordinate transformation, $r^2 = x^2 + y^2$

  => $x^2 + y^2 = (\rho - ex)^2 = \rho^2 - 2\rho ex + e^2x^2$

- Zeroing out the RHS gives...

$$x^2 + y^2 - (\rho^2 - 2\rho ex + e^2x^2) = 0$$

$$\Rightarrow \boxed{(1+e^2)x^2 + y^2 + 2\rho ex - \rho^2 = 0}$$ * All quadratic equations w/ two variables of this form are conic sections

* Special cases of our expression for conic sections written in terms of polar coordinates

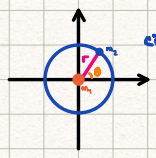
① $e=0$

$\Rightarrow r = \frac{p}{1+0\cos\theta} = \frac{p}{1} = p =$

$\Rightarrow \boxed{r = p = \frac{h^2}{\mu}}$

- The case where $e=0$ corresponds w/ the parameterization of a perfect circle w/ radius r:

* Note: Here, r is indp. of θ and is constant



circular orbit

② $0<e<1$ (majority of cases

- Example: $e=0.2$ (approximate eccentricity of Earth's orbit around the Sun
- Parameterizes an ellipse
  - $\theta=0$ corresponds w/...

$$r = \frac{p}{1+e\cos(0)} = \frac{p}{1+e}$$

$\Rightarrow$ Corresponding Cartesian coords.: $\left(\frac{p}{1+e}, 0\right)$

  - $\theta=\pi/2$ corresponds w/...

$$r = \frac{p}{1+e\cos(\pi/2)} = \frac{p}{1+0} = p$$

$\Rightarrow$ Corresponding Cartesian coords: $(0, p)$

  - $\theta=\pi$ corresponds w/...

$$r = \frac{p}{1+e\cos\pi} = \frac{p}{1-e}$$

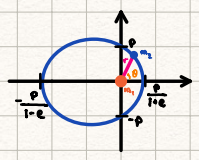
$\Rightarrow$ Corresponding Cartesian coords: $\left(-\frac{p}{1-e}, 0\right)$

  - $\theta=\frac{3\pi}{2}$ corresponds w/...

$$r = \frac{p}{1+e\cos(3\pi/2)} = \frac{p}{1-0} = p$$

$\Rightarrow$ Corresponding Cartesian coords.: $(0, -p)$

  - Overall path of m_2:



~ Importantly, the position of m_1 serves as one of the two foci
- The distance btwn m_1 and the position of m_2 @ $\theta=\pi/2$ & $\theta=3\pi/2$ is the semi-latus rectum (agrees w/ our def'n of p)

~ Since r varies w/ θ, the velocity of m_2 will also vary w/ θ

- Note: $r(\pi)$ is undefined for $e=1$
  - Thus, as $e\to1$, the shape of our elliptical path starts to on the LHS of m_2 starts "blowing up" to $r=\infty$
  - Also, $r(0)=p/2$ for $e=1$
- Note: $r(\pi/2)=r(3\pi/2)=p$ $\forall$ values of $0\le e\le1$

③ $e=1$

- This is the parabolic case
- In order to visualize, we need to use our quadratic eqn. for conic sections (i.e., Cartesian coords.):

$$(1+e^2)x^2 + y^2 + 2pex - p^2 = 0$$

  - Substituting in $e=1$ gives the general form for a parabolic equation:

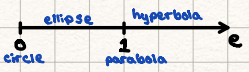
$$(1+1^2)x^2 + y^2 + 2p(1)x - p^2 = 0$$
$$2x^2 + y^2 + 2px - p^2 = 0$$
$$2x^2 + 2px + p^2 = -y^2$$

- Importantly, $e=1$ corresponds w/ some threshold energy where m_2 "escapes" orbit

④ $e>1$ $\Rightarrow$ parameterizes a hyperbola

* Cases of conic sections cont.

- Summary:

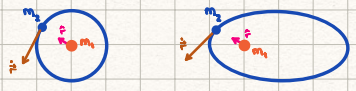


- $0 \le e < 1$ parameterizes a closed path
  - ~ Corresponds wl m_1 having negative energy
- $e = 1$ parameterizes a parabolic curve
  - ~ Corresponds wl m_1 having zero energy
- $e > 0$ parameterizes a hyperbolic curve
  - ~ Corresponds wl m_1 having positive energy

* Will discuss these energies next video

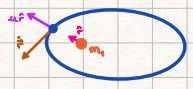
* More on the Laplace vector $\vec{c} = H\dot{\hat{r}} - \mu\hat{r}$

- We've already shown how to use $\vec{c}$ for solving Kepler problems
- Now lets consider what $\vec{c}$ looks like on our orbital diagrams
  - Diagrams of orbitals:



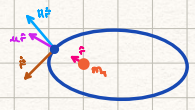
* Importantly, $\hat{r}$ points from m_1 to m_2 in both cases

- Earlier we showed that $\vec{c}$ is a conserved vector during the motion of m_2
  - ~ The $\mu\hat{r}$-term in $\vec{c}$ always points in the same direction as $\hat{r}$



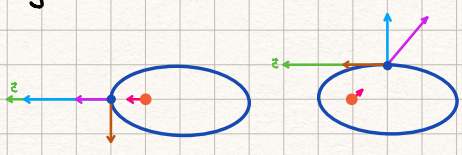
*Not drawn to scale

  - ~ The $H\dot{\hat{r}}$-term in $\vec{c}$ always points in a direction that corresponds wl a 90° clockwise rotation of $\dot{\hat{r}}$



*Not drawn to scale

* Note: $\dot{\hat{r}} \perp \hat{r}$ for $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$
  - ~ For these values of θ $H\dot{\hat{r}}$ ends up being parallel to $\hat{r}$
    - => The difference btwn $H\dot{\hat{r}}$ and $\mu\hat{r}$ will also be parallel to $\hat{r}$ (i.e., point in a horizontal direction)
      - => Since $\vec{c}$ is conserved, it will point horizontally $\forall\ \theta$
  - ~ Diagrams:

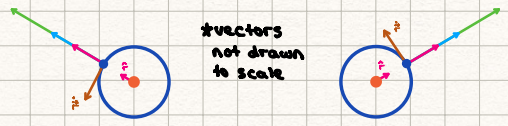


* $H\dot{\hat{r}} - \mu\hat{r}$ must always return a vector that is completely horizontal

* Note: $\vec{c}$ points in the direction at which m_2 is closest to m_1
  - ~ Officially, we say that $\vec{c}$ points in the direction of the major axis for our ellipse towards the point at which m_2 most closely approaches m_1 (i.e., the periapsis)

* Last Page cont.

- Now, let's think about $\vec{c}$ in the context of a perfectly circular orbit (i.e., $e=0$)
- Diagram of two possible positions



⇒ The only way $\vec{c}$ ends up being conserved is setting $\vec{c}=\vec{0}$

⇒ This corresponds nicely w/ the fact that $e=0$ for a perfectly circular orbit

*Recall:

$$e=\frac{|\vec{c}|}{\mu}=\frac{c}{\mu} \quad \Rightarrow \quad c=e\mu$$

• Can convert into a vector eqn. by introducing an eccentricity vector $\vec{e}$ where $|\vec{e}|=e$

$$\Rightarrow \vec{c}=\mu\vec{e}$$

- Write $\vec{c}=H\hat{h}-\mu\hat{r}$ in terms of $\vec{e}$ gives...

$$\mu\vec{e}=H\hat{h}-\mu\hat{r}$$

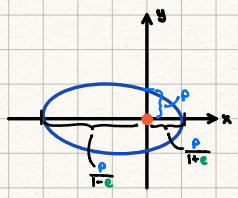
$$\Rightarrow \vec{e}=\frac{1}{\mu}H\hat{h}-\hat{r} = \text{a scalar multiple of the Laplace vector}$$

• Since $\vec{e}$ is a scalar multiple of $\vec{c}$, it is also a conserved vector of motion

Geometric Algebra - Kepler Problem (Part 3)

8/20/19

* Overview of Topics Covered:
  - Kepler's 3rd Law
  - Energy analysis
  - Vis Viva Equation
  - Virial Theorem
* Last Video:



$\ddot{\vec{r}} = -\frac{\mu}{r^2}\hat{r}$ (inverse-square law)

$r = \frac{h^2/\mu}{1+e\cos\theta} = \frac{\rho}{1+e\cos\theta}$

(radius of a given orbit has the form of a conic section)

* we get different types of conic sections for certain e-values
  - Recall: e gives the eccentricity of a conic section
    ρ is a parameter of a conic section called the semi-latus rectum (2ρ gives the magnitude of the "full" latus rectum

* Length of the major axis of a given ellipse can be determined using...

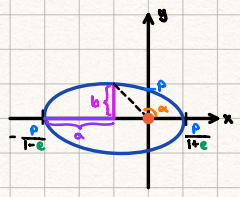
$$\frac{\rho}{1-e}+\frac{\rho}{1+e} = \frac{\rho(1+e)}{(1-e)(1+e)}+\frac{\rho(1-e)}{(1+e)(1-e)} = \frac{\rho[(1+e)+(1-e)]}{(1+e)(1-e)} = \frac{2\rho}{(1+e)(1-e)} = \frac{2\rho}{1-e^2}$$

$$\Rightarrow \boxed{\frac{2\rho}{1-e^2} = \text{distance of the major axis}}$$

- The semi-major axis (which we will denote as a) is half the length of the major axis

$$\Rightarrow \boxed{a = \frac{\rho}{1-e^2}}$$

* The minor axis is the distance btwn the maximum y-value and minimum y-value for our ellipse:
  - Let b denote the distance of the semi-minor axis (which is half the distance of the entire minor axis)
    - It turns out that $b = \frac{\rho}{\sqrt{1-e^2}}$ and $\underbrace{\alpha = \cos^{-1}(-e)}$ can use this relation and some trig. to derive our expression for b



* So far we have defined four parameters for our ellipse:
  1. e = eccentricity
  2. ρ = semi-latus rectum
  3. a = semi-major axis
  4. b = semi-minor axis

  } defined in terms of e

- These parameters will simplify the derivation of Kepler's 3rd Law
- Aside: Rewriting our a and b expressions give...

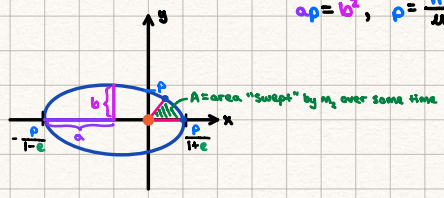
$$1-e^2 = \frac{\rho}{a} \quad \text{and} \quad \sqrt{1-e^2} = \frac{\rho}{b} \Rightarrow 1-e^2 = \frac{\rho^2}{b^2}$$

$$\Rightarrow 1-e^2 = \frac{\rho}{a} = \frac{\rho^2}{b^2} \Rightarrow \frac{\rho}{a} = \frac{\rho^2}{b^2} \Rightarrow \frac{1}{a} = \frac{\rho}{b^2} \Rightarrow \boxed{a\rho = b^2}$$

* relates 2nd, 3rd, & 4th parameters of an ellipse

Derivation of Kepler's 3rd Law

$ap = b^2$, $p = \frac{h^2}{\mu}$



- Kepler's 2nd Law is summarized by the following equation:

$\frac{dA}{dt} = \frac{1}{2}h$ (the area "swept" by m_2 as it orbits m_1 is equal to a constant)

*Recall: Last video, we defined h as the angular momentum per unit mass for m_2

- Q: How much area is "swept" out by m_2 over one period?
  - => Geometric relations tell us this area is equal to πab
    - => Combining w/ Kepler's 2nd Law gives...

$\pi ab = \frac{1}{2}hT$ where T denotes the period of a single orbit

- Multiplying both sides by 2 and squaring both sides gives...

$2\pi ab = hT \Rightarrow 4\pi^2 a^2 b^2 = h^2 T^2$

=> Squaring both sides allows us to make the following substitutions:

① $b^2 = ap$

$\Rightarrow 4\pi^2 a^2 (ap) = 4\pi^2 a^3 p = h^2 T^2$

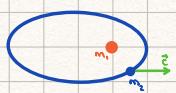
② $p = \frac{h^2}{\mu} \Rightarrow h^2 = p\mu$

$\Rightarrow 4\pi^2 a^3 \cancel{p} = \cancel{p}\mu T^2 \Rightarrow 4\pi^2 a^3 = \mu T^2$

=> Final Result: $\boxed{T^2 = \frac{4\pi^2}{\mu} a^3}$ Kepler's 3rd Law (the period squared is proportional to the cube of the semi-major axis => $T^2 \propto a^3$)

Energy analysis of different types of orbitals

- Consider the following system:



$\vec{c} = H\hat{r} - \mu\hat{r} = \mu\vec{e}$ Laplace Vector = a conserved vector that points along the major axis towards the periapsis (see last video)

$r = \frac{\frac{h^2}{\mu}}{1 + e\cos\theta}$

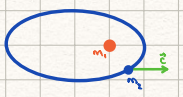
- It turns out that we can also write $e\cos\theta = \vec{e} \cdot \hat{r}$ where $\vec{e}$ is the eccentricity vector and a scalar multiple of $\vec{c}$

$\Rightarrow r = \frac{\frac{h^2}{\mu}}{1 + \vec{e}\cdot\hat{r}} \Rightarrow 1 + \vec{e}\cdot\hat{r} = \frac{\frac{h^2}{\mu}}{r} = \frac{h^2}{\mu r} \Rightarrow \boxed{\vec{e}\cdot\hat{r} = \frac{h^2}{\mu r} - 1}$ (we will use this eqn. soon)

- Overview of terms that define $\vec{c} = \mu\vec{e}$:
  - H = angular momentum bivector
  - $\dot{\vec{r}}$ = velocity vector
  - μ = gravitational parameter
  - $\hat{r}$ = unit displacement vector

* Energy analysis of different types of orbitals cont.

- Last Page:



$$\vec{c} = H\hat{r} - \mu\hat{r} = \mu\vec{e}\,; \quad r = \frac{\frac{h^2}{\mu}}{1+e\cos\theta}\,; \quad \vec{e}\cdot\hat{r} = \frac{h^2}{\mu r} - 1$$

- Now, let's rewrite $H\hat{r} - \mu\hat{r} = \mu\vec{e}$ $(= \vec{c})$ as follows:

$$H\hat{r} = \mu\vec{e} + \mu\hat{r} = \mu(\vec{e} + \hat{r})$$

- Squaring both sides gives...

*Note: $|\hat{r}| = 1$ and $|\hat{r}| = 1$

$$(H\hat{r})^2 = \mu^2(\vec{e}+\hat{r})^2$$
$$h^2v^2 = \mu^2(\vec{e}^2 + \vec{e}\hat{r} + \hat{r}\vec{e} + \hat{r}^2)$$
$$= \mu^2(|\vec{e}|^2 + \vec{e}\hat{r} + \hat{r}\vec{e} + |\hat{r}|^2)$$
$$= \mu^2(e^2 + \vec{e}\hat{r} + \hat{r}\vec{e} + 1)$$

*Recall: $\vec{u}\cdot\vec{v} = \frac{1}{2}(\vec{u}\vec{v} + \vec{v}\vec{u})$

$$\Rightarrow \vec{e}\hat{r} + \hat{r}\vec{e} = 2(\vec{e}\cdot\hat{r})$$

$$\Rightarrow h^2v^2 = \mu^2(e^2 + 2\vec{e}\cdot\hat{r} + 1)$$
$$= \mu^2\left[e^2 + 2\left(\frac{h^2}{\mu r} - 1\right) + 1\right]$$
$$= \mu^2\left(e^2 + \frac{2h^2}{\mu r} - 2 + 1\right)$$
$$= \mu^2\left(e^2 + \frac{2h^2}{\mu r} - 1\right)$$
$$h^2v^2 = \mu^2e^2 - 2h^2\frac{\mu}{r} - \mu^2$$

- Rearrange again s.t. our μ^2 terms are the only ones on the RHS:

$$h^2v^2 + 2h^2\frac{\mu}{r} = \mu^2e^2 - \mu^2 = \mu^2(e^2 - 1)$$

- Divide both sides by h^2:

$$v^2 + 2\frac{\mu}{r} = \frac{\mu^2}{h^2}(e^2 - 1)$$

*Note: v^2 is equivalent to $2\times$ the kinetic energy term and $2\frac{\mu}{r}$ is equivalent to $2\times$ the gravitational potential energy term from our eqn. for the total energy per unit mass

=> Thus, $v^2 + 2\frac{\mu}{r}$ is equivalent to $2T + 2U = 2(T+U) = 2E$

$$\Rightarrow 2E = \frac{\mu^2}{h^2}(e^2 - 1) = -\frac{\mu^2}{h^2}(1 - e^2)$$

$$\Rightarrow \boxed{E = -\frac{\mu^2}{2h^2}(1 - e^2)}$$

* Now let's evaluate how the total energy depends on the eccentricity

- Last Page: Derived an expression for the energy per unit mass

$$E = -\frac{\mu^2}{2h^2}(1-e^2)$$

- Recall: We get different types of orbits/conic sections for different cases of eccentricity values (see last video)

- Let's evaluate E for each case:

① Circle, $e=0$

$$\Rightarrow E = -\frac{\mu^2}{2h^2}(1-0^2) = -\frac{\mu^2}{2h^2} \Rightarrow E = -\frac{\mu^2}{2h^2} \Rightarrow E<0$$

*Both of the bounded/closed orbits are associated w/ negative energy

② Ellipse, $0<e<1$

$$E = -\frac{\mu^2}{2h^2}(1-e^2)$$

$e^2<1 \Rightarrow 1-e^2>0 \Rightarrow E<0$

③ Parabola, $e=1$

$$E = -\frac{\mu^2}{2h^2}(1-e^2) = -\frac{\mu^2}{2h^2}(1-1^2) = -\frac{\mu^2}{2h^2}(0) = 0 \Rightarrow E=0$$

*Both of the unbounded orbits are associated w/ nonegative energy

•Corresponds w/ some threshold where m_2 escapes orbit

④ Hyperbola, $e>1$

$$E = -\frac{\mu^2}{2h^2}(1-e^2)$$

$e^2>1 \Rightarrow 1-e^2<0 \Rightarrow E>0$

* Let's play around w/ our energy eqn. to make it more compact

$$E = -\frac{\mu^2}{2h^2}(1-e^2)$$

- Use $p = \frac{h^2}{\mu}$ to substitute $h^2 = \mu p$ into our energy eqn.:

$$E = -\frac{\mu^2}{2\mu p}(1-e^2) = -\frac{\mu}{2p}(1-e^2)$$

- Similarly, use $p = a(1-e^2)$ to substitute $\frac{1-e^2}{p} = \frac{1}{a}$ into our energy equation:

$$\Rightarrow \boxed{E = -\frac{\mu}{2a}}$$

*Recall: a denotes the length of the semi-major axis

Vis Viva Equation

- Last Page:
  - Derived an expression for the **total energy** as a function of the gravitational parameter μ and **semi-major axis** a

$$E = -\frac{\mu}{2a}$$ *Note: This gives the **total energy** per **unit mass**

- Also recall that the **total energy** E is equal to the sum of **kinetic** and **potential energies**

$$\Rightarrow E = -\frac{\mu}{2a} = T + U$$ * Here, T and U are the **kinetic** and **potential energies** per **unit mass**

$$\Rightarrow T = \frac{1}{2}v^2 \text{ and } U = -\frac{\mu}{r}$$

$$\Rightarrow \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$$

- Now we are going to solve the Vis Viva equation in terms of v^2:

$$\frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a} \Rightarrow \frac{1}{2}v^2 = \frac{\mu}{r} - \frac{\mu}{2a} \Rightarrow v^2 = \frac{2\mu}{r} - \frac{\mu}{a} = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$\Rightarrow \boxed{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$ **Vis Viva Equation**

- One feature of the Vis Viva eqn. is it provides a way to determine the **escape velocity**
  - Note: As an **object** approaches a ellipse → parabola "bifurcation", the **semi-major axis** approaches **infinity**

$$\Rightarrow a \to \infty \Rightarrow \frac{1}{a} \to 0$$

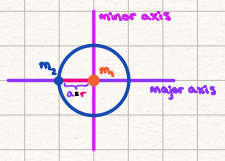
  - Therefore, we can write an expression for the **escape velocity** v_e as...

$$v_e^2 = \frac{2\mu}{r} \Rightarrow \boxed{v_e = \sqrt{\frac{2\mu}{r}}}$$

*Recall: v_e is the **velocity** that m_2 must obtain in order for its **kinetic energy** to surpass its **potential energy**

Virial Theorem

- Consider the case where we have a **perfectly circular orbit** (i.e., $e=0$)
  - Here, the **length of the semi-major axis** (a) is equal to the **radius** r of the **orbit**



$$\Rightarrow E = \frac{-\mu}{2a} = -\frac{\mu}{2r}$$ *Note: r is **constant**

  - Since the **potential energy** per **unit mass** can be expressed as $U = -\frac{\mu}{r}$, we can write...

$$E = \frac{1}{2}U$$

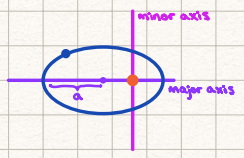
  - Combining w/ $E = T + U$ gives...

$$\frac{1}{2}U = T + U \Rightarrow T = \frac{1}{2}U - U = -\frac{1}{2}U \Rightarrow \boxed{T = -\frac{U}{2}}$$ **Virial Theorem**

*only applies for **circular orbits**

- Now, let's consider the case where we have an **elliptical orbit**
  - Here, the Virial Theorem gives the following relation

$$\boxed{\langle T \rangle = -\frac{1}{2}\langle U \rangle}$$ where $\langle\ \rangle$ denotes a **time-average**



Virial Theorem cont.

- Example: Consider a circular orbit that has $U = -10\ {}^{[\text{energy}]}/_{[\text{mass}]}$
  - Find the total energy:

$$E = \frac{1}{2}U = \frac{1}{2}\left(-10\ {}^{[\text{energy}]}/_{[\text{mass}]}\right) = -5\ {}^{[\text{energy}]}/_{[\text{mass}]}$$

  - Find the kinetic energy:

$$T = E - U = -5\ {}^{[\text{energy}]}/_{[\text{mass}]} - \left(-10\ {}^{[\text{energy}]}/_{[\text{mass}]}\right) = 5\ {}^{[\text{energy}]}/_{[\text{mass}]}$$

- Virial Theorem also shows up in other applications